

9. cvičení - řešení

Připomeňte si vzorce pro derivace: <https://www2.karlin.mff.cuni.cz/kuncova/materialy/deriv-t.pdf>.

Příklad 1 (a) $\int \sin^2 x + \cos^2 x dx = \int 1 dx \stackrel{c}{=} x$ (Pythagorova věta)

Jelikož $x' = 1$, tak $\int 1 dx \stackrel{c}{=} x$.

Příklad 1 (b) $\int 5x^7 + \frac{9}{x^2} dx \stackrel{\text{lin.}}{=} 5 \int x^7 dx + 9 \int x^{-2} dx \stackrel{c}{=} 5 \frac{x^8}{8} + 9 \frac{x^{-1}}{-1} = \frac{5}{8}x^8 - \frac{9}{x}$

Jelikož $(x^n)' = n \cdot x^{n-1}$ pro $n \neq -1$, tak $\int x^n dx = \frac{1}{n+1}x^{n+1}$ pro $n \neq -1$.

Příklad 1 (c) $\int \sqrt{x} + e^{2x} dx \stackrel{\text{lin.}}{=} \int x^{\frac{1}{2}} dx + \int e^{2x} dx \stackrel{c}{=} \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2}e^{2x} = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}e^{2x}$

Platí, že $(\frac{1}{2}e^{2x})' = \frac{1}{2} \cdot 2 \cdot e^{2x} = e^{2x}$. Tedy skutečně $\int e^{2x} dx \stackrel{c}{=} \frac{1}{2}e^{2x}$.

Příklad 1 (d) $\int 2 \sin 3x + e^{-x} + 4 dx \stackrel{\text{lin.}}{=} 2 \int \sin 3x dx + \int e^{-x} dx + 4 \int 1 dx \stackrel{c}{=} \frac{-2}{3} \cos 3x - e^{-x} + 4x$

Opět zřejmě $(\frac{-2}{3} \cos 3x)' = \frac{-2}{3} \cdot (-\sin 3x) \cdot 3 = 2 \sin 3x$. Pak tedy platí, že $\int 2 \sin 3x dx \stackrel{c}{=} \frac{-2}{3} \cos 3x$.

Příklad 1 (e) $\int x^{\frac{1}{5}} - \frac{2}{x^3} dx \stackrel{c}{=} \frac{x^{\frac{1}{5}+1}}{\frac{1}{5}+1} - 2 \frac{x^{-3+1}}{-3+1} = \frac{5}{6}x^{\frac{6}{5}} + x^{-2}$

Příklad 1 (f) $\int (x+1)^2 + \cos \frac{x}{2} + \frac{7}{x} dx \stackrel{\text{lin.}}{=} \int x^2 + 2x + 1 dx + 2 \sin \frac{x}{2} + 7 \log |x| \stackrel{c}{=} \frac{1}{3}x^3 + x^2 + x + 2 \sin \frac{x}{2} + 7 \log |x|$

Příklad 1 (g) $\int (2-x)^4 - \frac{1}{1+x^2} dx = \int (x-2)^4 - \frac{1}{1+x^2} dx \stackrel{c}{=} \frac{(x-2)^5}{5} - \arctan x$

Příklad 1 (h)

$$\begin{aligned} \int \frac{x^3 + 4x + 1}{2\sqrt{x}} dx &\stackrel{\text{lin.}}{=} \frac{1}{2} \int x^{3-\frac{1}{2}} dx + \frac{4}{2} \int x^{1-\frac{1}{2}} dx + \frac{1}{2} \int x^{-\frac{1}{2}} dx = \\ &\stackrel{c}{=} \frac{1}{2} \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + 2 \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{1}{2} \frac{x^{\frac{-1}{2}+1}}{\frac{-1}{2}+1} = \frac{1}{2} \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + 2 \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{2} \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = \\ &= \frac{1}{7}x^{\frac{7}{2}} + \frac{4}{3}x^{\frac{3}{2}} + \sqrt{x} \end{aligned}$$

Příklad 1 (i) $\int \frac{1}{\sqrt{1-x^2}} dx \stackrel{c}{=} \arcsin x$

Příklad 2 (a)

$$\begin{aligned} \int 2xe^{-x} dx &\stackrel{\text{lin.}}{=} 2 \int x \cdot e^{-x} dx \stackrel{\text{PP}}{=} |u = x, u' = 1, v' = e^{-x}, v = -e^{-x}| = \\ &= 2(-xe^{-x}) - \int 1 \cdot (-e^{-x}) dx = -2xe^{-x} + \int e^{-x} dx \stackrel{c}{=} -2xe^{-x} - e^{-x} \end{aligned}$$

Příklad 2 (b)

$$\begin{aligned} \int x \sin x dx &\stackrel{\text{PP}}{=} |u = x, u' = 1, v = -\cos x, v' = \sin x| = -x \cos x - \int 1 \cdot (-\cos x) dx = \\ &\stackrel{c}{=} \sin x - x \cos x \end{aligned}$$

Příklad 2 (c)

$$\begin{aligned}
& \int (3x^3 + x^2 + 1)e^{3x} dx \stackrel{\text{PP}}{=} \left| u = 3x^3 + x^2 + 1, u' = 9x^2 + 2x, v = \frac{1}{3}e^{3x}, v' = e^{3x} \right| = \\
& = (3x^3 + x^2 + 1)e^{3x} \frac{1}{3} - \frac{1}{3} \int (9x^2 + 2x)e^{3x} dx \stackrel{\text{PP}}{=} \left| u = 9x^2 + 2x, u' = 18x + 2, v = \frac{1}{3}e^{3x}, v' = e^{3x} \right| = \\
& = (3x^3 + x^2 + 1)e^{3x} \frac{1}{3} - \frac{1}{3} \left(\frac{1}{3}(9x^2 + 2x)e^{3x} - \frac{1}{3} \int (18x + 2)e^{3x} dx \right) = \\
& \stackrel{\text{PP}}{=} \left| u = 18x + 2, u' = 18, v = \frac{1}{3}e^{3x}, v' = e^{3x} \right| = \\
& = (3x^3 + x^2 + 1) \frac{e^{3x}}{3} - \frac{e^{3x}}{9}(9x^2 + 2x) + \frac{1}{9} \left(\frac{1}{3}(18x + 2)e^{3x} - \frac{18}{3} \int e^{3x} dx \right) = \\
& \stackrel{c}{=} (3x^3 + x^2 + 1) \frac{e^{3x}}{3} - \frac{e^{3x}}{9}(9x^2 + 2x) + \frac{e^{3x}}{27}(18x + 2) - \frac{2}{3} \cdot \frac{1}{3}e^{3x} = \\
& = e^{3x} \left(x^3 + \frac{x^2}{3} + \frac{1}{3} - x^2 - \frac{2}{9}x + \frac{2}{3}x + \frac{2}{27} - \frac{2}{9} \right) = e^{3x} \left(x^3 - \frac{2}{3}x^2 + \frac{4}{9}x + \frac{5}{27} \right)
\end{aligned}$$

Příklad 2 (d)

$$\begin{aligned}
& \int e^x \cos x dx \stackrel{\text{PP}}{=} \left| u = \cos x, u' = -\sin x, v = e^x, v' = e^x \right| = e^x \cos x + \int e^x \sin x dx = \\
& \stackrel{\text{PP}}{=} \left| u = \sin x, u' = \cos x, v = e^x, v' = e^x \right| = e^x \cos x + e^x \sin x - \int e^x \cos x dx
\end{aligned}$$

Máme:

$$\int e^x \cos x dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx \implies 2 \int e^x \cos x dx \stackrel{c}{=} e^x (\sin x + \cos x)$$

Platí tedy, že $\int e^x \cos x dx \stackrel{c}{=} \frac{1}{2}e^x (\sin x + \cos x)$

Příklad 2 (e)

$$\begin{aligned}
& \int x \arctan x dx \stackrel{\text{PP}}{=} \left| u = \arctan x, u' = \frac{1}{1+x^2}, v = \frac{x^2}{2}, v' = x \right| = \frac{x^2}{2} \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} dx = \\
& = \frac{x^2}{2} \arctan x - \frac{1}{2} \int 1 - \frac{1}{1+x^2} dx \stackrel{c}{=} \frac{x^2}{2} \arctan x - \frac{1}{2} (x - \arctan x)
\end{aligned}$$

Příklad 2 (f)

$$\int x^2 \log x dx \stackrel{\text{PP}}{=} \left| u = \log x, u' = \frac{1}{x}, v = \frac{x^3}{3}, v' = x^2 \right| = \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 dx \stackrel{c}{=} \frac{x^3}{3} \log x - \frac{x^3}{9}$$

Příklad 2 (g)

$$\begin{aligned}
\int \sqrt{x} \log^2 x dx &\stackrel{\text{PP}}{=} \left| u = \log^2 x, u' = 2 \log x \cdot \frac{1}{x}, v = \frac{2}{3} x^{\frac{3}{2}}, v' = \sqrt{x} \right| = \frac{2}{3} \sqrt{x^3} \log^2 x - \frac{4}{3} \int \log x \cdot x^{\frac{1}{2}} dx = \\
&\stackrel{\text{PP}}{=} \left| u = \log x, u' = \frac{1}{x}, v = \frac{2}{3} x^{\frac{3}{2}}, v' = \sqrt{x} \right| = \frac{2}{3} \sqrt{x^3} \log^2 x - \frac{4}{3} \left(\frac{2}{3} x^{\frac{3}{2}} \log x - \frac{2}{3} \int x^{\frac{1}{2}} dx \right) = \\
&\stackrel{\text{c}}{=} \frac{2}{3} \sqrt{x^3} \log^2 x - \frac{8}{9} x^{\frac{5}{2}} \log x + \frac{16}{27} x^{\frac{3}{2}}
\end{aligned}$$

Příklad 2 (h) $\int \log^2 x + \log x^2 dx$

$$\begin{aligned}
\int \log^2 x dx &\stackrel{\text{PP}}{=} \left| u = \log^2 x, u' = 2 \log x \frac{1}{x}, v = x, v' = 1 \right| = x \log^2 x - 2 \int \log x dx = \\
&\stackrel{\text{PP}}{=} \left| u = \log x, u' = \frac{1}{x}, v = x, v' = 1 \right| = x \log^2 x - 2x \log x + 2 \int 1 dx \stackrel{\text{c}}{=} x \log^2 x - 2x \log x + 2x
\end{aligned}$$

$$\int \log x^2 dx \stackrel{\text{PP}}{=} \left| u = \log x^2, u' = \frac{2x}{x^2} = \frac{2}{x}, v = x, v' = 1 \right| = x \log x^2 - 2 \int x \frac{1}{x} dx \stackrel{\text{c}}{=} x \log x^2 - 2x$$

$$\begin{aligned}
\int \log^2 x + \log x^2 dx &\stackrel{\text{lin.}}{=} \int \log^2 x dx + \int \log x^2 dx \stackrel{\text{c}}{=} x \log^2 x - 2x \log x + 2x + x \log x^2 - 2x = \\
&= x \log^2 x - x \log x^2 + x \log x^2 = x \log^2 x
\end{aligned}$$

Příklad 2 (i)

$$\begin{aligned}
\int \sin(\log 2x) dx &\stackrel{\text{PP}}{=} \left| u = \sin(\log 2x), u' = \cos(\log 2x) \frac{1}{x}, v = x, v' = 1 \right| = \\
&= x \sin(\log 2x) - \int x \frac{1}{x} \cos(\log 2x) dx \stackrel{\text{PP}}{=} \left| u = \cos(\log 2x), u' = -\sin(\log 2x) \frac{1}{x}, v = x, v' = 1 \right| = \\
&= x \sin(\log 2x) - x \cos(\log 2x) - \int x \sin(\log 2x) \frac{1}{x} dx = x \sin(\log 2x) - x \cos(\log 2x) - \int \sin(\log 2x) dx
\end{aligned}$$

Máme tedy:

$$\begin{aligned}
\int \sin(\log 2x) dx &= x \sin(\log 2x) - x \cos(\log 2x) - \int \sin(\log 2x) dx \implies \\
&\implies \int \sin(\log 2x) dx \stackrel{\text{c}}{=} \frac{1}{2} x (\sin(\log 2x) - \cos(\log 2x))
\end{aligned}$$

Příklad 3(a)

$$\int (3x - 2)^6 dx = |y = 3x - 2, dy = 3dx| = \int y^6 \frac{1}{3} dy \stackrel{\text{c}}{=} \frac{1}{3} \frac{y^7}{7} = \frac{(3x - 2)^7}{21}$$

Poznámka: v substituci máme, že $dy = 3dx$. To vzniklo zderivováním rovnice $y = 3x - 2$. Ze vztahu $dy = 3dx$ plyne: $dx = \frac{1}{3}dy$, což se objevuje v integrálu po provedení substituce.

Příklad 3(b)

$$\int \sin(2x+1)dx = |y = 2x+1, dy = 2dx| = \int \sin y \cdot \frac{1}{2}dx \stackrel{c}{=} -\frac{1}{2}\cos y = -\frac{1}{2}\cos(2x+1)$$

Příklad 3(c)

$$\int \frac{1}{(x+1)^2+1}dx = |y = x+1, dy = dx| = \int \frac{1}{y^2+1}dy \stackrel{c}{=} \arctan y = \arctan(x+1)$$

Příklad 3(d)

$$\int \frac{x}{(x^2+1)^2}dx = |y = x^2+1, dy = 2xdx| = \int \frac{1}{2y^2}dy \stackrel{c}{=} \frac{1}{2} \cdot \frac{-1}{y} = -\frac{1}{2(x^2+1)}$$

Příklad 3(e)

Využijeme vztahu: $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

$$\begin{aligned} \int -\frac{1}{\sqrt{8-3x^2}}dx &\stackrel{\text{lin.}}{=} \frac{-1}{\sqrt{8}} \int \frac{1}{\sqrt{1-\frac{3}{8}x^2}}dx = \left| y = \sqrt{\frac{3}{8}}x, dy = \sqrt{\frac{3}{8}}dx \right| \stackrel{\text{lin.}}{=} \frac{-1}{\sqrt{8}} \cdot \frac{\sqrt{8}}{\sqrt{3}} \int \frac{1}{\sqrt{1-y^2}}dy = \\ &\stackrel{c}{=} -\frac{1}{\sqrt{3}} \arcsin y = -\frac{1}{\sqrt{3}} \arcsin \left(\sqrt{\frac{3}{8}}x \right) \end{aligned}$$

Příklad 3(f)

$$\int 3xe^{-x^2}dx = |y = -x^2, dy = -2dx| = \int \frac{-3}{2}e^y dy \stackrel{c}{=} \frac{-3}{2}e^y = -\frac{3}{2}e^{-x^2}$$

Příklad 3(g)

$$\int \frac{1}{(1+x^2)\arctan^3 x}dx = \left| y = \arctan x, dy = \frac{1}{1+x^2}dx \right| = \int \frac{1}{y^3}dy \stackrel{c}{=} \frac{-1}{2y^2} = -\frac{1}{2\arctan^2 x}$$

Příklad 3(h)

$$\begin{aligned} \int \frac{1}{1+\sqrt{x-1}}dx &= \left| y = \sqrt{x-1}, dy = \frac{1}{2\sqrt{x-1}}dx \right| = \int \frac{2y}{y+1}dy \stackrel{\text{lin.}}{=} 2 \int \frac{y+1-1}{y+1}dy = \\ &= 2 \int 1 - \frac{1}{y+1}dy \stackrel{\text{lin.}}{=} 2y - 2 \int \frac{1}{y+1}dy = |z = y+1, dz = dy| = 2y - 2 \int \frac{1}{z}dz \stackrel{c}{=} 2y - 2 \log |z| = \\ &= 2\sqrt{x-1} - 2 \log (1 + \sqrt{x-1}) \end{aligned}$$

Pozn.: absolutní hodnota není potřeba, neb $1 + \sqrt{x-1} > 0$, je-li výraz definován.

Příklad 3(i)

Uvědomme si: $\tan^5 x = \frac{\sin^5 x}{\cos^5 x}$ a $\sin^4 x = (\sin^2 x)^2 = (1 - \cos^2 x)^2$.

$$\begin{aligned} \int \tan^5 x dx &= |y = \cos x, dy = -\sin x dx| = \int -\frac{(1-y^2)^2}{y^5} dy = \int -\frac{1-2y^2+y^4}{y^5} dy = \\ &= \int -\frac{1}{y^5} + 2\frac{1}{y^3} - \frac{1}{y} dy \stackrel{c}{=} \frac{1}{4y^4} - \frac{1}{y^2} - \log|y| = \frac{1}{4\cos^4 x} - \frac{1}{\cos^2 x} - \log|\cos x| \end{aligned}$$

Příklad 5(a)

$$\int \frac{1}{x^6} \sin \frac{1}{x^5} dx = \left| y = \frac{1}{x^5}, dy = \frac{-5}{x^6} \right| \stackrel{\text{lin.}}{=} \frac{-1}{5} \int \sin y dy \stackrel{c}{=} \frac{1}{5} \cos y = \frac{1}{5} \cos \frac{1}{x^5}$$

Příklad 5(b)

$$\int x^3 \log 2x dx \stackrel{\text{PP}}{=} \left| u = \log 2x, u' = \frac{1}{x}, v = \frac{x^4}{4}, v' = x^3 \right| = \frac{x^4}{4} \log 2x - \int \frac{x^4}{4} \frac{1}{x} dx \stackrel{c}{=} \frac{x^4}{4} \log 2x - \frac{x^4}{16}$$

Příklad 5(c)

$$\begin{aligned} \int \left(1 - \frac{1}{x^2}\right) \sqrt{x\sqrt{x}} dx &\stackrel{\text{lin.}}{=} \int \sqrt{x\sqrt{x}} dx - \int \frac{\sqrt{x\sqrt{x}}}{x^2} dx = \int x^{\frac{3}{4}} dx - \int \frac{x^{\frac{3}{4}}}{x^2} dx = \\ &= \frac{4}{7} x^{\frac{7}{4}} - \int x^{-\frac{5}{4}} dx \stackrel{c}{=} \frac{4}{7} x^{\frac{7}{4}} + 4x^{-\frac{1}{4}} \end{aligned}$$

Příklad 5(d)

Nejdříve spočteme: $\int 2xe^{-x^2} dx = |y = -x^2, dy = -2x dx| = -\int e^y dy = -e^y = -e^{-x^2}$

$$\begin{aligned} \int x^5 e^{-x^2} dx &= \int \frac{-1}{2} x^4 \cdot (-2x) e^{-x^2} dx \stackrel{\text{PP}}{=} \left| u = \frac{-1}{2} x^4, u' = -2x^3, v = e^{-x^2}, v' = (-2x) e^{-x^2} \right| = \\ &= \frac{-1}{2} x^4 e^{-x^2} - \int e^{-x^2} (-2)x \cdot x^2 dx = |y = -x^2, dy = -2x dx| = -\frac{1}{2} x^4 e^{-x^2} - \int e^y (-y) dy = \\ &\stackrel{\text{lin.}}{=} -\frac{1}{2} x^4 e^{-x^2} + \int y e^y dy \stackrel{\text{PP}}{=} |u = y, u' = 1, v = e^y, v' = e^y| = -\frac{1}{2} x^4 e^{-x^2} + y e^y - \int e^y dy = \\ &\stackrel{c}{=} -\frac{1}{2} x^4 e^{-x^2} + y e^y - e^y = -\frac{1}{2} x^4 e^{-x^2} + (-x^2) e^{-x^2} - e^{-x^2} \end{aligned}$$